LAU
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Lebanese American University

Review for final

MTH 304 - Spring 2014

1) Discuss the equilibrium solutions for $\frac{dy}{dt} = y^2 - \alpha^2 y + 4 = 0$.

Plot the phase lines in the following cases: $\alpha = \pm 3$; $\alpha = \pm 2$; $\alpha = 0$.

Solution: ~~Characteristic equation~~ Equilibrium solutions are solutions to: $y^2 - \alpha y + 4 = 0$.

$\Delta = \alpha^4 - 16 > 0$ if $\alpha \in (-\infty, -2) \cup (+2, \infty)$; $\Delta = 0$ if $\alpha = \pm 2$, and

$\Delta < 0$ if $\alpha \in (-2, 2)$

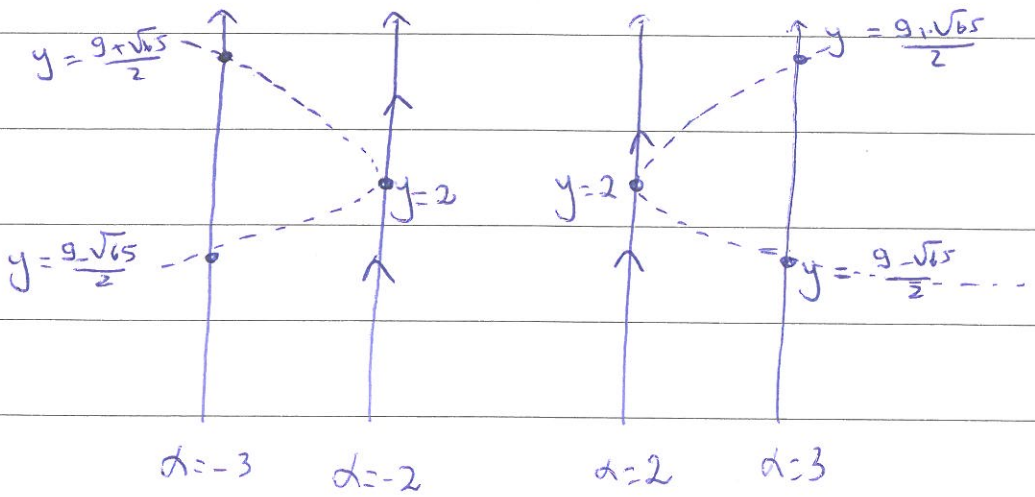
for $\alpha = -3$, we have 2 equilibrium: $y = \frac{+\alpha^2 \pm \sqrt{\Delta}}{2} = \frac{+9 \pm \sqrt{65}}{2}$

for $\alpha = -2$, we have 1 eq: $y = \frac{\alpha^2}{2} = \frac{4}{2} = 2$

for $\alpha = +2$, we have 1 as well: $y = \frac{\alpha^2}{2} = \frac{4}{2} = 2$.

for $\alpha = 3$, we have 2: $y = \frac{\alpha^2 \pm \sqrt{\Delta}}{2} = \frac{9 \pm \sqrt{65}}{2}$.

Phase lines



2) find the general solution to: $y'' + 5y' + 4y = 2e^{-t}$.

Solution: Characteristic is : $r^2 + 5r + 4 = 0 \Rightarrow (r+4)(r+1) = 0$

$\therefore r = -1$ and $r = 4$

$\therefore y_1 = e^{-t}$ and $y_2 = e^{4t}$.

A guess for the particular solution for the non-homog. equation is: $y_p = Ate^{-t}$ (Ae^{-t} does not work).

$y_p' = Ae^{-t} - Ate^{-t}$; $y_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$.

Substitute : $-2Ae^{-t} + \underline{Ate^{-t}} + 5Ae^{-t} - \underline{5Ate^{-t}} + \underline{4Ate^{-t}} = 2e^{-t}$

$\therefore \quad 3Ae^{-t} = 2e^{-t} \Rightarrow \boxed{A = \frac{2}{3}}$

Hence, general solution is: $y = k_1 e^{-t} + k_2 e^{4t} + \frac{2}{3} t e^{-t}$.

3) Calculate : $L^{-1}\left[\frac{3s}{s^2+4s+5}\right]$.

Solution: $\frac{3s}{s^2+4s+5} = \frac{3s}{(s^2+4s+4)+1} = \frac{3s}{(s+2)^2+1}$

$= 3\,\frac{(s+2)-2}{(s+2)^2+1} = 3 \cdot \frac{s+2}{(s+2)^2+1} - 6 \cdot \frac{1}{(s+2)^2+1}$

$\therefore L^{-1}\left[\frac{3s}{s^2+4s+5}\right] = 3L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 6L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= 3e^{-2t}\cos t - 6e^{-2t}\sin t$.

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④(a) Convert the following 2^{nd}-order equation into a system, then find the solution curves to that system:

$$y'' + 9y = 0$$

<u>Solution</u>: Let $v = \frac{dy}{dt} \Rightarrow \frac{dv}{dt} = -9y$.

$\therefore$ the system is: $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -9y \end{cases}$ $\therefore$ matrix is $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$.

Eigenvalues of A: $\lambda^2 - tr(A)\lambda + \det(A) = 0$

$$\Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i.$$

$\lambda = 3i \rightarrow A\vec{v} = \lambda\vec{v}$ implies $\begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (3i)\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{cases} y = 3ix \\ -9x = 3iy \rightarrow y = \frac{-9x}{3i} = 3ix \end{cases}$ $\therefore \vec{v} = \langle 1, 3i \rangle$ is one eigenvector.

A complex solution is then: $e^{(3i)t} \begin{bmatrix} 1 \\ 3i \end{bmatrix} = (\cos 3t + i\sin 3t)\begin{bmatrix} 1 \\ 3i \end{bmatrix}$

$$= \begin{bmatrix} \cos(3t) + i\sin(3t) \\ -3\sin(3t) + 3i\cos(3t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(3t) \\ -3\sin(3t) \end{bmatrix}} + i \underbrace{\begin{bmatrix} \sin(3t) \\ 3\cos(3t) \end{bmatrix}}$$

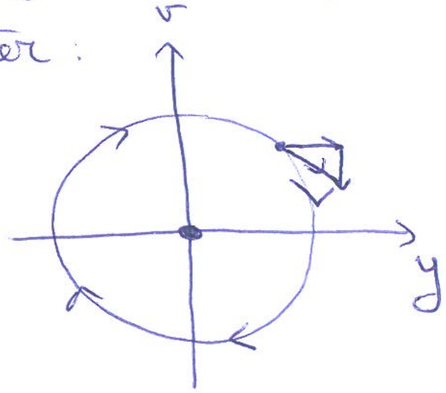
$\therefore$ General solution is: $\begin{bmatrix} y \\ v \end{bmatrix} = k_1 \begin{bmatrix} \cos(3t) \\ -3\sin(3t) \end{bmatrix} + k_2 \begin{bmatrix} \sin(3t) \\ 3\cos(3t) \end{bmatrix}$.

$\therefore\ y(t) = k_1 \cos(3t) + k_2 \sin(3t).$

$v(t) = -3k_1 \sin(3t) + 3k_2 \cos(3t).$

b) Discuss the phase plane of the system.

Solution: Since the eigenvalue is pure imaginary, then $(0,0)$ is a center:



5) Consider $y'' - 4y' + 3y = 0;\ y(0) = 1;\ y'(0) = 2.$

(a) find the solution to this ode.

Solution: $r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0 \begin{cases} r = 3 \\ r = 1 \end{cases}$

$\therefore\ y = k_1 e^{3t} + k_2 e^{t}$

But $y(0) = 1 \Rightarrow k_1 + k_2 = 1 \rightarrow k_2 = 1 - k_1$

$y'(0) = 2 \Rightarrow 3k_1 + k_2 = 0$

$\Rightarrow 3k_1 + 1 - k_1 = 0 \Rightarrow \boxed{2k_1 = 1}$ and $\boxed{k_2 = \frac{1}{2}}$.

$\boxed{k_1 = \frac{1}{2}}$

(b) If $\sum_{n=0}^{\infty} a_n x^n$ is a series solution for this ODE, prove that the recurrence relation satisfies: $a_{n+2} = \dfrac{4a_{n+1}}{(n+2)} - \dfrac{3a_n}{(n+1)(n+2)}$

for $n = 0, 1, 2, 3, \ldots$

Solution: $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$; $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Substitute:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4 \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow (n+2)(n+1) a_{n+2} - 4(n+1) a_{n+1} + 3 a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{4(n+1) a_{n+1}}{(n+1)(n+2)} - \frac{3 a_n}{(n+1)(n+2)} = \frac{4 a_{n+1}}{n+2} - \frac{3 a_n}{(n+1)(n+2)}$$

(c) We can show that $a_n = \frac{3^n + 1}{2(n!)}$; $n = 0, 1, 2, \ldots$ (do not show)
Check that the series solution agrees with the solution found in (a).

Solution: $y = \sum_{n=0}^{\infty} \left(\frac{3^n + 1}{2(n!)}\right) x^n = \sum \frac{3^n x^n}{2(n!)} + \sum \frac{x^n}{2(n!)}$

$$= \frac{1}{2} \sum \frac{(3x)^n}{n!} + \frac{1}{2} \sum \frac{x^n}{n!} = \frac{1}{2} e^{3x} + \frac{1}{2} e^x.$$

6) find the general solution to: $y'' + 16y = \sec(4t)$

Solution: $r^2 + 16 = 0 \Rightarrow r = \pm 4i \Rightarrow y_1 = \cos(4t)$ and $y_2 = \sin(4t)$

are 2 solutions to the homog. equation.

for the non-homog., $y_p = v_1 y_1 + v_2 y_2$, where

$$v_1 = -\int \frac{y_2 \cdot (\sec(4t))}{W[y_1, y_2]}\, dt = -\int \frac{\sin(4t) \cdot \frac{1}{\cos(4t)}}{4}\, dt$$

$$= -\frac{1}{4}\int \frac{\sin(4t)}{\cos(4t)} = \frac{1}{16} \ln|\cos(4t)|$$

and $$v_2 = \int \frac{y_1 \cdot \sec(4t)}{W[y_1, y_2]}\, dt = \int \frac{\cos(4t) \cdot \frac{1}{\cos(4t)}}{4}\, dt = \frac{1}{4} t.$$

7) Brine containing 1lb/gal of salt is poured at the rate of 1 gal/min into a tank that initially contained 100gl of pure water. The mixture is drained off at the rate of 2 gal/min.

(a) Write a first-order ode (an IVP) that describes the rate of change of salt in tank.

Solution: $s(t)$ = amount of salt in tank at time t.

$s(0) = 0$ (pure water) and $\frac{ds}{dt} = 1 \times 1 - 2 \times \frac{s}{100 - t}$

$$\Rightarrow \frac{ds}{dt} + \left(\frac{2}{100-t}\right) s = 1.$$

This is a linear ODE : $\mu(t) = e^{\int \frac{2}{100-t} dt} = e^{-2\ln|100-t|} = \frac{1}{(100-t)^2}$

$\therefore s(t) = (100-t)^2 \int \frac{1}{(100-t)^2} dt = (100-t)^2 \left[\frac{1}{100-t} + c \right]$

$\therefore s(t) = (100-t) + c(100-t)^2$; But $s(0) = 0 \Rightarrow$, therefore,

$0 = 100 + c(100)^2 \Rightarrow c = \frac{1}{100}$.

$\therefore s(t) = (100-t) + \frac{(100-t)^2}{100}$.

⑧ Compute $L[u_2(t)e^{3t}]$ directly from the definition of the laplace transform.

Solution : $L[u_2(t)e^{3t}] = \int_0^\infty u_2(t)e^{3t} \cdot e^{-st} dt$

$$= \int_2^\infty e^{3t} \cdot e^{-st} dt = \int_2^\infty e^{(3-s)t} dt$$

$$\therefore L[u_2(t)e^{3t}] = \lim_{l \to \infty} \int_2^l e^{(3-s)t} dt$$

$$= \lim_{l \to \infty} \left. \frac{e^{(3-s)t}}{3-s} \right|_2^l$$

$$= \frac{e^{-2(s-3)}}{s-3} - \lim_{l \to \infty} \left[\cancel{\frac{e^{-l(s-3)}}{s-3}} \right] \to 0 \text{ if } s > 3$$

$$\therefore L[u_2(t)e^{3t}] = \frac{e^{-2(s-3)}}{s-3} \quad \text{if } s > 3.$$

This is a linear ODE: $\mu(t) = e^{+\int \frac{1}{100-t}\,dt} = e^{-\ln|100-t|} = \frac{1}{(100-t)}$

$\therefore s(t) = \frac{(100-t)}{\cancel{1}} \int \frac{1}{(100-t)}\,dt = \frac{1}{100-t}\left[100t - t^2 + c\right] = \frac{t(100-t)}{100-t} + \frac{c}{100-t}$

$\therefore s(t) = t + \frac{c}{100-t}$; but $s(0) = 0 \Rightarrow 0 = \frac{c}{100}$